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The radiation and scattering of surface waves by vertical barriers

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A train of small-amplitude surface waves is incident normally on an arbitrary arrangement of thin barriers lying in a vertical plane in deep water. Each barrier is allowed to make small rolling or swaying oscillations of the same frequency as that of the incident wave. The boundary-value problem for the consequent fluid motion, assumed two-dimensional, is solved exactly using a technique which enables the amplitudes of the scattered waves far from the barriers to be readily determined. Reference is made to the associated wave radiation problem and to the calculation of forces and moments on the barriers.

1. Introduction

The two-dimensional scattering and radiation of surface waves by various configurations of thin barriers lying in a vertical plane in deep water have been investigated by several authors. The case of a submerged semi-infinite barrier was considered by Dean (1945) and Ursell (1947). Various aspects of a single surface-piercing barrier have been treated by Haskind (1948, 1959), Levine & Rodemich (1958) and Ursell (1947, 1948). Evans (1970) solved the problem for a single submerged oscillating barrier and Porter (1972) investigated the transmission of waves through a gap in a semi-infinite barrier. Two authors have considered the general problem of an arbitrary number of barriers: Lewin (1963) the scattering problem and Mei (1966) associated initial-value problems.

Problems of this type are generally solved by the use of a reduction technique. Either the so-called reduced potential is invoked at the outset, or a formulation is adopted which leads to an integral equation. In the latter case, the reduction manifests itself in the form of a differential operator which must be applied to the integral equation to render it amenable to standard solution techniques. Whichever of these two basic approaches is employed, it is ultimately necessary to establish that the solution of the reduced problem does in fact satisfy the problem as posed in its unreduced form. In the case of the integral-equation formulation, for instance, the equivalence of the original equation and its reduced counterpart must be demonstrated. The solution of the reduced problem contains arbitrary constants which are, in fact, determined in the process of satisfying the original problem. In practice, the determination of constants in this way involves a considerable amount of detailed manipulation, especially for more complicated barrier arrangements. Thus, quantities of prime physical interest, the transmission and reflexion coefficients, are not easily found.

In this paper, an alternative to the reduction process is presented. It has the advantage that the amplitudes of the scattered waves are found with a minimum of detailed analysis. This is achieved because a standard type of singular integral equation is obtained directly, no arbitrary constants being generated in its solution. Rather, certain solvability conditions must be complied with and these, together with a set of requirements at the barrier edges, serve to determine all unknown quantities.

The basis of the technique is taken from a paper by Williams (1966), who used it in rather a different manner to solve the scattering problem for a fixed surface-piercing barrier. Williams made use of properties of a weakly singular Volterra integral equation, this approach being particularly suited to the case of a single barrier.

The problem posed is the general one of a wave train incident on an arbitrary arrangement of barriers, which are allowed to perform small rolling or swaying oscillations. Thus, all the configurations previously considered are included here, and the transmission and reflexion coefficients for several of these are calculated by way of illustration. A solution for the case of wave radiation by the moving barriers in otherwise still water is an immediate consequence, and the calculation of wave forces and moments on the barriers is mentioned, including use of the Haskind relations.

2. Formulation

Perfect fluid in irrotational motion occupies the region $y \geqslant 0$, the x axis lying in the undisturbed free surface. A train of small-amplitude surface waves of angular frequency σ is supposed to be travelling from $x = -\infty$. The motion of the fluid can be described by the velocity potential

$$\Phi(x, y, t) = \mathscr{R}\{e^{-i\sigma t}\phi(x, y)\},$$

where, making the usual assumptions of linearized theory, $\phi(x, y)$ must satisfy

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0, \quad y \geqslant 0, \tag{2.1}$$

$$\partial\phi/\partial y + \alpha\phi = 0, \quad y = 0, \quad \alpha = \sigma^2/g, \tag{2.2}$$

$$|\nabla\phi| \to 0, \quad y \to \infty. \tag{2.3}$$

It is remarked that, although the effect of surface tension is omitted in the present formulation, an extension of the ensuing method can be developed to incorporate it.

In the absence of obstacles in the fluid, it is convenient to take the solution for $\phi(x, y)$ as

$$\phi(x, y) = I\phi_0(x, y), \quad I = -iga/\sigma, \quad \phi_0(x, y) = e^{i\alpha x - \alpha y},$$

corresponding to the incident elevation

$$\eta(x, t) = a\cos(\alpha x - \sigma t).$$

Suppose now that the wave train is partially reflected by n thin barriers lying in $x = 0$ and occupying the intervals B_j: $a_j \leqslant y \leqslant b_j$ $(j = 1, 2, \ldots, n)$. Each barrier

B_j makes small rolling oscillations of angular amplitude ω_j and angular frequency σ about a fixed point c_j. The normal velocity of a point on the barrier is given by

$$\partial\Phi/\partial x = \sigma\omega_j(y-c_j)\cos(\sigma t+\epsilon_j), \quad y\in B_j,$$

the linearization being valid provided that $\omega_j \ll 1$ and that $\max|(y-c_j)\,\omega_j|$ is $O(a)$ for $y\in B_j$. The constant ϵ_j is included to allow the barrier to oscillate with an arbitrary phase. The case of small translational oscillations (sway) of the barrier B_j is achieved by allowing $c_j\to\infty$ and $\omega_j\to 0$ in such a way that $c_j\omega_j$ remains finite.

Inclusion of the barriers therefore requires that

$$\partial\phi/\partial x = \sigma\omega_j(y-c_j)\,e^{-i\epsilon_j}, \quad x = 0\pm, \quad y\in B_j \quad (j = 1,2,\ldots n), \tag{2.4}$$

together with the radiation condition, which implies that

$$\phi(x,y) \sim \begin{Bmatrix} I\phi_0(x,y)+R\phi_0(-x,y), & x\to-\infty, \\ T\phi_0(x,y), & x\to+\infty. \end{Bmatrix} \tag{2.5}$$

Here R and T are the unknown constants we are primarily seeking. The function ϕ and its first derivatives must be continuous across $x = 0$, $y\in G$, where G denotes the aggregate of the gaps between the barriers. That is,

$$G = \bigcup_{j=1}^{n+1} G_j,$$

where

$$G_1 = (0,a_1), \quad G_{n+1} = (b_n,\infty) \quad \text{and} \quad G_{j+1} = (b_j,a_{j+1}) \quad \text{for} \quad j = 1,2,\ldots,n-1.$$

It is well known that in problems of this type the fluid velocities necessarily possess integrable singularities at the ends of the barriers. Specifically,

$$\partial\phi/\partial r \sim r^{-\gamma}, \quad r\to 0, \quad 0<\gamma<1, \tag{2.6}$$

where r denotes the distance from a point in the fluid to any of the edges.

We now consider ϕ as consisting of two parts: the first representing the solution when a fixed barrier occupies $x = 0$, $y\geqslant 0$ (the incident wave being totally reflected) and the second accounting for the presence of gaps in this barrier. Continuity of the normal fluid velocity across $x = 0$ indicates that this second part must be an odd function of x. (See Lamb (1932, p. 517), where this device is used in a similar acoustic problem.)

Thus we write

$$\phi(x,y) = \begin{Bmatrix} I\phi_0(x,y)+I\phi_0(-x,y)+\hat{\phi}(x,y), & x<0, \\ -\hat{\phi}(-x,y), & x>0, \end{Bmatrix} \tag{2.7}$$

and seek the solution for $\hat{\phi}(x,y)$ in $x<0$. Obviously $\hat{\phi}$ must satisfy (2.1) off the barriers, (2.2), (2.3) and

$$\partial\hat{\phi}/\partial x = \sigma\omega_j(y-c_j)\,e^{-i\epsilon_j}, \quad x = 0, \quad y\in B_j \quad (j = 1,2,\ldots,n). \tag{2.8}$$

Reference to (2.5) reveals that we require

$$\hat{\phi}(x,y) \sim -T\phi_0(-x,y), \quad x \to -\infty, \tag{2.9}$$

and that $R+T = I$. The continuity of $\partial\phi/\partial x$ across $x = 0$ for $y \in G$ is guaranteed by (2.7), and ϕ and $\partial\phi/\partial y$ are continuous there if

$$\hat{\phi}(0,y) = -I\phi_0(0,y), \quad y \in G. \tag{2.10}$$

Invoking the idea implied by Williams (1966), we introduce a function $\psi(x,y)$ defined by

$$\partial\psi/\partial y - \alpha\psi = T\phi_0(-x,y) + \hat{\phi}(x,y), \quad x \leqslant 0. \tag{2.11}$$

It follows that (2.9) is satisfied if

$$\psi,\ \partial\psi/\partial y \to 0, \quad x \to -\infty. \tag{2.12}$$

Further, if

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0, \quad x < 0, \quad y \geqslant 0, \tag{2.13}$$

then $\hat{\phi}$ is also harmonic in that region. The free-surface condition (2.2) on $\hat{\phi}$ then gives, on using (2.12),

$$\psi(x,0) = 0, \quad x \leqslant 0. \tag{2.14}$$

Equations (2.8) and (2.10) reveal that we require

$$\partial\psi(0,y)/\partial x = f_j(y) = F_j e^{\alpha y} + \tfrac{1}{2}iT\, e^{-\alpha y} - (\sigma\omega_j/\alpha^2)\, e^{-i\epsilon_j}\{(y-c_j)\,\alpha + 1\},$$
$$y \in B_j \quad (j = 1,2,\ldots,n), \tag{2.15}$$

$$\psi(0,y) = h_j(y) = H_j e^{\alpha y} + (2\alpha)^{-1}(I-T)\, e^{-\alpha y}, \quad y \in G_j \quad (j = 1,2,\ldots,n+1), \tag{2.16}$$

where F_j and H_j are constants of integration. To satisfy (2.3) we impose

$$|\nabla\psi| \to 0, \quad y \to \infty, \tag{2.17}$$

and note that this implies that $H_{n+1} = 0$, and hence that

$$\psi(0,y) \sim e^{-\alpha y}, \quad y \to \infty. \tag{2.18}$$

An important property of the function ψ, revealed by (2.6) and (2.11), is that its first derivatives must be bounded at the ends of the barriers. Also, it is to be noted that $\psi(0,y)$ and $\partial\psi(0,y)/\partial y$ must be continuous across the ends of the barriers in order that $\phi(0,y)$ (and hence the first-order pressure) also has this property. Thus, the expressions for ψ on the barriers, when evaluated at their ends, must be compatible with the corresponding values given by (2.16),

$$\psi(0,a_j) = h_j(a_j), \quad \psi(0,b_j) = h_{j+1}(b_j) \quad (j = 1,2,\ldots,n). \tag{2.19}$$

It is evident from the foregoing formulation that the requirements defining ψ are sufficient to generate a velocity potential ϕ which satisfies all the prescribed conditions. As was remarked by Williams in his particular problem, a proof that the conditions on ψ are also necessary is superfluous, since the posed boundary-value problem for ϕ has a unique solution (see John 1948).

3. Determination of $\psi(x, y)$

We determine ψ by constructing an integral equation readily solvable by standard techniques. To this end, we note that an appropriate Green's function is

$$G(x,y|\xi,\eta) = \hat{G}(x,y|\xi,\eta)+\hat{G}(x,y|-\xi,-\eta)-\hat{G}(x,y|-\xi,\eta)-\hat{G}(x,y|\xi,-\eta),$$

where $\hat{G}(x,y|\xi,\eta) = (-1/2\pi)\log\{(x-\xi)^2+(y-\eta)^2\}^{\frac{1}{2}}$. In particular

$$G(x,y|\xi,0) = G(x,y|0,\eta) = 0.$$

Application of Green's theorem to the function G and ψ in $x \leqslant 0$, $y \geqslant 0$ yields

$$\psi(x,y) = -\int_0^\infty \frac{\partial G}{\partial \xi}(x,y|0,\eta)\,\psi(0,\eta)\,d\eta, \tag{3.1}$$

whence
$$\frac{\partial \psi}{\partial x}(0,y) = \frac{1}{\pi}\frac{d^2}{dy^2}\int_0^\infty \log\left|\frac{y-\eta}{y+\eta}\right|\psi(0,\eta)\,d\eta.$$

Integration by parts, using (2.14), (2.18) and the fact that $\psi(0,\eta)$ is everywhere continuous, easily leads to

$$\frac{\partial \psi}{\partial x}(0,y) = \frac{1}{\pi}\int_0^\infty \frac{2y}{\eta^2-y^2}\frac{\partial \psi}{\partial \eta}(0,\eta)\,d\eta,$$

in which the integral is to be interpreted as a Cauchy principal value, as are subsequent singular integrals. Thus we obtain the singular integral equation

$$\frac{1}{\pi}\int_B \frac{2y}{\eta^2-y^2}\frac{\partial \psi}{\partial \eta}(0,\eta)\,d\eta = F(y), \quad y \in B = \bigcup_{j=1}^{n} B_j, \tag{3.2}$$

where
$$F(y) = \frac{1}{\pi}\int_G \frac{2yh'(\eta)\,d\eta}{y^2-\eta^2} - f(y), \quad y \in B, \tag{3.3}$$

$$f(y) = f_j(y), \quad y \in B_j; \quad h(y) = h_j(y), \quad y \in G_j.$$

The solution of (3.2) for $\partial\psi(0,\eta)/\partial\eta$, required to be bounded at all ends of B, is obtained using the method expounded by Muskhelishvili (1963). Thus, an appropriate Carlemann function is introduced and use is made of the Plemelj formulae to deduce an equivalent Riemann–Hilbert problem, the details involved being similar to those in Porter (1972). The solution of the Riemann–Hilbert problem can be determined by reference to Muskhelishvili.

We note at this point that there are three special cases of the problem posed hereto. Although the basic steps in the determination of ψ are the same in all these cases, each requires individual attention in matters of detail and for clarity we cite the cases separately.

Case 1. *The problem as posed: n submerged barriers, each of finite length.* Since $\psi(x,y)$ must satisfy (2.14) and (2.17) we have

$$H_1 = -(I-T)/2\alpha, \quad H_{n+1} = 0.$$

The remaining unknown constants are $F_1, F_2, \ldots, F_n$, $H_2, \ldots, H_n$ and T. The appropriate solution of (3.2), namely

$$\frac{\partial \psi}{\partial y}(0,y) = \frac{2y}{\pi}(R_1(y))^{\frac{1}{2}}\int_B \frac{F(\eta)\,d\eta}{(R_1(\eta))^{\frac{1}{2}}(y^2-\eta^2)}, \quad y \in B, \tag{3.4}$$

exists provided that the so-called solvability conditions

$$\int_B \frac{\eta^{2k} F(\eta)\, d\eta}{(R_1(\eta))^{\frac{1}{2}}} = 0 \quad (k = 0, 1, \ldots, n-1), \tag{3.5}$$

are satisfied. These n conditions are a direct consequence of the fact that the solution is bounded at all ends of B. In (3.4) and (3.5) we have written

$$R_1(y) = \prod_{p=1}^{n} (y^2 - a_p^2)(y^2 - b_p^2).$$

The meaning of $(R_1(y))^{\frac{1}{2}}$ accords with the convention adopted by Muskhelishvili (*loc. cit.*). Thus, introducing the complex variable $\zeta = y + i\nu$ we define

$$(R_1(y))^{\frac{1}{2}} = \lim_{\nu \to 0+} (R_1(\zeta))^{\frac{1}{2}},$$

where $(R_1(\zeta))^{\frac{1}{2}}$ is understood to be that branch which is analytic in the ζ plane cut along $-b_j \leqslant y \leqslant -a_j$ and $a_j \leqslant y \leqslant b_j$ $(j = 1, 2, \ldots, n)$ and for which

$$(R_1(\zeta))^{\frac{1}{2}} \sim |\zeta|^{2n} \quad \text{as} \quad |\zeta| \to \infty.$$

The same meaning will be attached to similar branch functions occurring hereafter.

On inserting (3.3) into (3.4), we note that it is permissible to reverse the order of the resulting repeated integral. Further, contour integration and use of the Plemelj formulae reveal that

$$\int_B \frac{2\eta\, d\eta}{(R_1(\eta))^{\frac{1}{2}} (y^2 - \eta^2)} = \begin{cases} -\pi i/(R_1(y))^{\frac{1}{2}}, & y \in G, \\ 0, & y \in B. \end{cases}$$

Hence

$$\frac{\partial \psi}{\partial y}(0, y) = \frac{2y}{\pi} (R_1(y))^{\frac{1}{2}} \left\{ i \int_G \frac{h'(\eta)\, d\eta}{(R_1(\eta))^{\frac{1}{2}} (y^2 - \eta^2)} - \int_B \frac{f(\eta)\, d\eta}{(R_1(\eta))^{\frac{1}{2}} (y^2 - \eta^2)} \right\}, \quad y \in B. \tag{3.6}$$

Similar manipulation reduces (3.5) to

$$i \int_G \frac{\eta^{2k} h'(\eta)\, d\eta}{(R_1(\eta))^{\frac{1}{2}}} - \int_B \frac{\eta^{2k} f(\eta)\, d\eta}{(R_1(\eta))^{\frac{1}{2}}} = 0 \quad (k = 0, 1, \ldots, n-1). \tag{3.7}$$

Thus, the value of $\psi(0, y)$ on each barrier is found by integrating (3.6), and making use of the end conditions (2.19). In particular, we must have

$$h_{k+1}(b_k) - h_k(a_k) = \frac{2}{\pi} \int_{a_k}^{b_k} y (R_1, (y))^{\frac{1}{2}}\, dy \left\{ i \int_G \frac{h'(\eta)\, d\eta}{(R_1(\eta))^{\frac{1}{2}} (y^2 - \eta^2)} - \int_B \frac{f(\eta)\, d\eta}{(R_1(\eta))^{\frac{1}{2}} (y^2 - \eta^2)} \right\}$$
$$(k = 1, 2, \ldots, n). \tag{3.8}$$

The requirements (3.7) and (3.8) together constitute $2n$ equations in the $2n$ unknown constants listed previously; in particular the quantities R and T are now determined in principle.

It can be shown that the n relations (3.8), added together, reduce to a further expression of the type (3.7) with $k = n$. Thus we may replace (3.7) and (3.8) by the $2n$ equivalent requirements

$$\left.\begin{aligned} &i\int_G \frac{\eta^{2k} h'(\eta)\, d\eta}{(R_1(\eta))^{\frac{1}{2}}} - \int_B \frac{\eta^{2k} f(\eta)\, d\eta}{(R_1(\eta))^{\frac{1}{2}}} = 0 \quad (k = 0, 1, \ldots, n), \\ &h_{k+1}(b_k) - h_k(a_k) = \frac{2}{\pi}\int_{a_k}^{b_k} y(R_1(y))^{\frac{1}{2}}\, dy \left\{ i\int_G \frac{h'(\eta)\, d\eta}{(R_1(\eta))^{\frac{1}{2}}(y^2-\eta^2)} \right. \\ &\qquad \left. - \int_B \frac{f(\eta)\, d\eta}{(R_1(\eta))^{\frac{1}{2}}(y^2-\eta^2)} \right\} \quad (k = 1, 2, \ldots, n-1). \end{aligned}\right\} \tag{3.9}$$

Case 2. *The uppermost barrier intersects the free surface*; *that is,* $a_1 = 0$ *and* G_1 *is omitted.* We assume that the fluid velocity is bounded at the point $(0, 0)$. Equations (2.14) and (2.17) give

$$F_1 = \sigma\omega_1 e^{-i\epsilon_1}(1-\alpha C_1)/\alpha^2 - \tfrac{1}{2}iT, \quad H_{n+1} = 0.$$

The complete solution for $\psi(0, y)$, $y \in B$, can be found either by applying the limiting process $a_1 \to 0$ to case 1, or by constructing the appropriate solution of (3.2) and proceeding as in case 1. By each method we find that

$$\frac{\partial\psi}{\partial y}(0, y) = \frac{2}{\pi}(R_2(y))^{\frac{1}{2}}\left\{ i\int_G \frac{\eta h'(\eta)\, d\eta}{(R_2(\eta))^{\frac{1}{2}}(y^2-\eta^2)} - \int_B \frac{\eta f(\eta)\, d\eta}{(R_1(\eta))^{\frac{1}{2}}(y^2-\eta^2)} \right\}, \quad y \in B, \tag{3.10}$$

where $R_2(y) = R_1(y)/(y^2 - a_1^2)$. The constants $F_2, \ldots, F_n$, $H_2, \ldots, H_n$ and T are determined from

$$\left.\begin{aligned} &i\int_G \frac{\eta^{2k-1} h'(\eta)\, d\eta}{(R_2(\eta))^{\frac{1}{2}}} - \int_B \frac{\eta^{2k-1} f(\eta)\, d\eta}{(R_2(\eta))^{\frac{1}{2}}} = 0 \quad (k = 1, 2, \ldots, n), \\ &h_{k+1}(b_k) - h(a_k) = \frac{2}{\pi}\int_{a_k}^{b_k} (R_2(y))^{\frac{1}{2}}\, dy \left\{ i\int_G \frac{\eta h'(\eta)\, d\eta}{(R_2(\eta))^{\frac{1}{2}}(y^2-\eta^2)} \right. \\ &\qquad \left. - \int_B \frac{\eta f(\eta)\, d\eta}{(R_2(\eta))^{\frac{1}{2}}(y^2-\eta^2)} \right\} \quad (k = 1, 2, \ldots, n-1), \end{aligned}\right\} \tag{3.11}$$

where $h_1(a_1) = 0$.

Case 3. *The deepest barrier extends infinitely far into the fluid*; $b_n = \infty$, $\omega_n = 0$, G_{n+1} *is omitted,* $H_1 = -(I-T)/2\alpha$ *and* $F_n = 0$. The solution of (3.2) must tend to zero as $y \to \infty$ on B_n. It can, after some manipulation, be written in the form

$$\frac{\partial\psi}{\partial y}(0, y) = \frac{2y}{\pi}(R_3(y))^{\frac{1}{2}}\left\{ i\int_G \frac{h'(\eta)\, d\eta}{(R_3(\eta))^{\frac{1}{2}}(y^2-\eta^2)} - \int_B \frac{f(\eta)\, d\eta}{(R_3(\eta))^{\frac{1}{2}}(y^2-\eta^2)} \right\}, \quad y \in B, \tag{3.12}$$

with solvability conditions

$$i\int_G \frac{\eta^{2k} h'(\eta)\, d\eta}{(R_3(\eta))^{\frac{1}{2}}} - \int_B \frac{\eta^{2k} f(\eta)\, d\eta}{(R_3(\eta))^{\frac{1}{2}}} = 0 \quad (k = 0, 1, \ldots, n-1), \tag{3.13}$$

where $R_3(y) = R_1(y)/(y^2 - b_n^2)$. By using (3.13), (3.12) can be written in an alternative form in which R_3 is replaced by R_b/R_a, where

$$R_a(y) = \prod_{p=1}^{n} (y^2 - a_p^2), \quad R_b(y) = R_3(y)/R_a(y).$$

The end conditions are

$$h_{k+1}(b_k)-h_k(a_k)=\frac{2}{\pi}\int_{a_k}^{b_k} y\,(R_3(y))^{\frac{1}{2}}\,dy\left\{i\int_G \frac{h'(\eta)\,d\eta}{(R_3(\eta))^{\frac{1}{2}}(y^2-\eta^2)}-\int_B \frac{f(\eta)\,d\eta}{(R_3(\eta))^{\frac{1}{2}}(y^2-\eta^2)}\right\}$$
$$(k=1,2,\ldots,n-1), \quad (3.14)$$

$$-h_n(a_n)=\frac{2}{\pi}\int_{a_n}^{\infty} y\left(\frac{R_b(y)}{R_a(y)}\right)^{\frac{1}{2}} dy\left\{i\int_G\left(\frac{R_a(\eta)}{R_b(\eta)}\right)^{\frac{1}{2}}\frac{h'(\eta)\,d\eta}{y^2-\eta^2}-\int_B\left(\frac{R_a(\eta)}{R_b(\eta)}\right)^{\frac{1}{2}}\frac{f(\eta)\,d\eta}{y^2-\eta^2}\right\}. \quad (3.15)$$

If these n relations are added we obtain, not a further condition as in cases 1 and 2, but an identity. We conclude that (3.15) is identically satisfied if (3.14) are satisfied. The unknown constants $F_1, \ldots, F_{n-1}, H_2, \ldots, H_n$ and T are found from (3.13) and (3.14).

Case 4. *The aggregate of cases* 2 *and* 3 *in which* $a_1=0$, $b_n=\infty$ *and* $\omega_n=0$. We omit G_1 and G_{n+1} and have

$$F_1=\sigma\omega_1 e^{-i\epsilon_1}(1-\alpha C_1)/\alpha^2-\tfrac{1}{2}iT, \quad F_n=0.$$

The solution for $\psi(0,y)$, $y\in B$, can be found along the lines indicated in case 3, or by taking the limit $a_1\to 0$ in that case. It is found that

$$\frac{\partial\psi}{\partial y}(0,y)=\frac{2}{\pi}(R_4(y))^{\frac{1}{2}}\left\{i\int_G\frac{\eta h'(\eta)\,d\eta}{(R_4(\eta))^{\frac{1}{2}}(y^2-\eta^2)}-\int_B\frac{\eta f(\eta)\,d\eta}{(R_4(\eta))^{\frac{1}{2}}(y^2-\eta^2)}\right\}, \quad y\in B, \quad (3.16)$$

where $R_4(y)=R_3(y)/(y^2-a_1^2)$. The constants $F_2, \ldots F_{n-1}, H_2, \ldots H_n$ and T are found from

$$\left.\begin{aligned} &i\int_G\frac{\eta^{2k-1}h'(\eta)\,d\eta}{(R_4(\eta))^{\frac{1}{2}}}-\int_B\frac{\eta^{2k-1}f(\eta)\,d\eta}{(R_4(\eta))^{\frac{1}{2}}}=0 \quad (k=1,2,\ldots,n-1),\\ &h_{k+1}(b_k)-h_k(a_k)=\frac{2}{\pi}\int_{a_k}^{b_k}(R_4(y))^{\frac{1}{2}}dy\left\{i\int_G\frac{\eta h'(\eta)\,d\eta}{(R_4(\eta))^{\frac{1}{2}}(y^2-\eta^2)}\right.\\ &\qquad\left.-\int_B\frac{\eta f(\eta)\,d\eta}{(R_4(\eta))^{\frac{1}{2}}(y^2-\eta^2)}\right\} \quad (k=1,2,\ldots,n-1),\end{aligned}\right\} \quad (3.17)$$

where $h_1(a_1)=0$.

4. Particular cases

The simplest examples of each of cases 1–4 have previously been examined, as noted in § 1. We briefly indicate the use of the results obtained in the previous section by reference to these particular problems.

(*a*) *A submerged fixed barrier* (*case* 1, $n=1$). Here

$$h_1(y)=-(I-T)\sinh(\alpha y)/\alpha, \quad h_2(y)=(I-T)\,e^{-\alpha y}/2\alpha,$$
$$f_1(y)=F_1e^{\alpha y}+\tfrac{1}{2}iTe^{-\alpha y}, \quad R_1(y)=(y^2-a_1^2)(y^2-b_1^2).$$

The two constants F_1 and T are determined by elementary manipulation from the relevant conditions, (3.9), namely

$$i\int_0^{a_1}\frac{\eta^{2k}h_1'(\eta)\,d\eta}{(R_1(\eta))^{\frac{1}{2}}}+i\int_{b_1}^{\infty}\frac{\eta^{2k}h_2'(\eta)\,d\eta}{(R_1(\eta))^{\frac{1}{2}}}-\int_{a_1}^{b_1}\frac{\eta^{2k}f_1(\eta)\,d\eta}{(R_1(\eta))^{\frac{1}{2}}}=0 \quad (k=0,1).$$

(*b*) *A surface-piercing fixed barrier* (*case* 2, $n = 1$). Here

$$f_1(y) = -iT\sinh(\alpha y), \quad h_2(y) = (I-T)\,e^{-\alpha y}/2\alpha.$$

The quantity T is found from the only appropriate condition (3.11),

$$\int_{b_1}^{\infty}\frac{\eta h_2'(\eta)\,d\eta}{(\eta^2-b_1^2)^{\frac{1}{2}}}+\int_0^{b_1}\frac{\eta f_1(\eta)\,d\eta}{(b_1^2-\eta^2)^{\frac{1}{2}}} = 0,$$

whence we directly obtain

$$T = IK_1(\alpha, b_1)/\{K_1(\alpha b_1)-\pi i I_1(\alpha b_1)\},$$

where I_1 and K_1 are modified Bessel functions of the first and second kinds respectively.

(*c*) *A submerged fixed semi-infinite barrier* (*case* 3, $n = 1$). In this case

$$f_1(y) = \tfrac{1}{2}iT\,e^{-\alpha y}, \quad h_1(y) = -(I-T)\sinh(\alpha y)/\alpha$$

and T is defined by

$$\int_0^{a_1}\frac{h_1'(\eta)\,d\eta}{(a_1^2-\eta^2)^{\frac{1}{2}}}-\int_{a_1}^{\infty}\frac{f_1(\eta)\,d\eta}{(\eta^2-a_1^2)^{\frac{1}{2}}} = 0;$$

that is,
$$T = I\pi I_0(\alpha a_1)/\{\pi I_0(\alpha a_1)-iK_0(\alpha a_1)\},$$

where I_0 and K_0 again denote modified Bessel functions.

(*d*) *A gap in a fixed semi-infinite barrier* (*case* 4, $n = 2$). Here

$$f_1(y) = -iT\sinh(\alpha y), \quad f_2(y) = \tfrac{1}{2}iT\,e^{-\alpha y}, \quad h_2(y) = H_2 e^{\alpha y}+(I-T)\,e^{-\alpha y}/2\alpha,$$
$$R_4(y) = (y^2-b_1^2)\,(y^2-a_2^2).$$

The values of H_2 and T are determined from the pair of simultaneous equations

$$i\int_{b_1}^{a_2}\frac{\eta h_2'(\eta)\,d\eta}{(R_4(\eta))^{\frac{1}{2}}}-\int_0^{b_1}\frac{\eta f_1(\eta)\,d\eta}{(R_4(\eta))^{\frac{1}{2}}}-\int_{a_2}^{\infty}\frac{\eta f_2(\eta)\,d\eta}{(R_4(\eta))^{\frac{1}{2}}} = 0,$$

$$h_2(b_1) = \frac{2}{\pi}\int_0^{b_1}(R_4(y))^{\frac{1}{2}}\,dy\left\{i\int_{b_1}^{a_2}\frac{\eta\,h_2'(\eta)\,d\eta}{(R_4(\eta))^{\frac{1}{2}}(y^2-\eta^2)}-\int_0^{b_1}\frac{\eta f_1(\eta)\,d\eta}{(R_4(\eta))^{\frac{1}{2}}(y^2-\eta^2)}\right.$$
$$\left.-\int_{a_2}^{\infty}\frac{\eta f_2(\eta)\,d\eta}{(R_4(\eta))^{\frac{1}{2}}(y^2-\eta^2)}\right\}.$$

In all instances, agreement is obtained with previous calculations.

5. Conclusion

In each of the cases, unique expressions for $\psi(0,y)$ and $\partial\psi(0,y)/\partial y$ are found on the barriers. The value of $\psi(x,y)$ off $x = 0$ can be found using (3.1). In particular, for the scattering problem posed, the amplitude and phase of the outgoing waves are determined via the solution of an explicit system of algebraic equations. The case of waves radiated by the rolling or swaying barriers in otherwise still fluid is obviously achieved by setting $I = 0$ ($R = -T$) throughout.

Since $\phi(0,y)$ is thus known on the barriers, the first-order and second-order mean (time-averaged over a wave period) horizontal forces and moments on the barriers can readily be calculated (see, for example, Evans (1970), in which such calculations are made for a single submerged barrier). It is of interest to note that, in case 1, the second-order mean horizontal force on the jth barrier assumes the particularly simple form

$$\bar{X}_j^{(2)} = \rho\alpha^2\mathscr{R}\{I(H_j - H_{j+1})\}.$$

Thus the net force on all the barriers is

$$\bar{X}^{(2)} = \sum_{j=1}^{n} \bar{X}_j^{(2)} = \rho\alpha^2\mathscr{R}\{I(H_1 - H_{n+1})\} = -\tfrac{1}{2}\rho\alpha\mathscr{R}(RI). \tag{5.1}$$

This result was obtained by Evans for the single barrier. It can be shown that (5.1) also holds for the three limiting versions of case 1.

We remark that knowledge of the far-field waves radiated by the barriers in states of sway and roll, readily found by the above method, are sufficient to calculate the net first-order horizontal force and net moment for cases 1 and 2. This is achieved by making use of the Haskind relations discussed in, for example, Newman (1972). Thus, for instance, it is a straightforward matter to produce these quantities for cases 1 and 2 with $n = 1$, agreement being obtained with Evans (1970) and Haskind (1959).

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